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LETTER TO THE EDITOR

A class of pseudoparticle solutions of the SO(5) Yang–Mills equations

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Abstract. Using γ-matrix representations of the SO(5) algebra, a special *ansatz* for the solutions of the Yang–Mills equations is tested. Non-trivial solutions are found, and their charges are computed.

Recently the pseudoparticle (Belavin *et al* 1975) solutions of the SU(2) Yang–Mills field equations attributed to t'Hooft (Jackiw *et al* 1977) have become well known. In this letter, we give a brief presentation of some pseudoparticle solutions of the SO(5) Yang–Mills equations. We recover from our results the SO(4) pseudoparticle solutions in the form obtained by J Madore *et al* (private communication).

Our starting point follows from a general *ansatz* made for the SO(4) Yang–Mills connection (the vector potential) in a previous article by one of us (Tchrakian 1977). The distinguishing feature of that *ansatz* was that it had the parity-definite form

$$A_\mu = \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\lambda]x_\lambda\left(\frac{1+\gamma_5}{2}\right)g_1(r) + \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\lambda]x_\lambda\left(\frac{1-\gamma_5}{2}\right)g_2(r). \tag{1}$$

The main consequence of (1) was that the self-duality of the resulting curvature $F_{\mu\nu}$, *and not its double self-duality*, gave rise to the BPST solutions, even though the γ-matrices used in (1) are generators of SO(4) and not SU(2). This was a small technical advantage.

We now generalise this *ansatz* by first, relaxing the restriction to spherical symmetry and second, by including other γ-matrix bases which together with the bases in (1) make up the basic representation for the SO(6) algebra,

$$\begin{aligned} &L_{\mu\nu} = \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\nu], \qquad \mu, \nu = 1, 2, 3, 4 \\ &L_{5\mu} = \gamma_\mu, \qquad L_{6\mu} = \mathrm{i}\gamma_5\gamma_\mu, \qquad L_{56} = \tfrac{1}{2}\gamma_5 \\ &\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}. \end{aligned} \tag{2}$$

The *ansatz* is

$$A_\mu = \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\lambda]\left(\frac{1+\gamma_5}{2}\right)\alpha_\lambda + \frac{\mathrm{i}}{4}[\gamma_\mu, \gamma_\lambda]\left(\frac{1-\gamma_5}{2}\right)\beta_\lambda + \frac{1}{4e}\gamma_\mu\Lambda + \frac{1}{4e}\mathrm{i}\gamma_5\gamma_\mu\Omega \tag{3}$$

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where

$$\alpha_\mu = a_\mu + b_\mu = -\frac{1}{e}\partial\mu \ln \theta$$
$$\beta_\mu = a_\mu - b_\mu = +\frac{1}{e}\partial\mu \ln \psi. \tag{4}$$

The resulting expression for the curvature then is

$$\begin{aligned}F_{\mu\nu} &= (\partial_\mu A_\nu + \mathrm{i}eA_\mu A_\nu) - [\mu, \nu] \\ &= \left[\frac{\mathrm{i}}{4}[\gamma_\nu, \gamma_\lambda]\left(\frac{1+\gamma_5}{2}\right)(\partial_\mu\alpha_\lambda + e\alpha_\mu\alpha_\lambda) + \frac{\mathrm{i}}{8}[\gamma_\mu, \gamma_\nu]\left(\frac{1+\gamma_5}{2}\right)e(\alpha^2 + \tfrac{1}{4}\Lambda^2 + \tfrac{1}{4}\Omega^2)\right. \\ &\quad + \frac{\mathrm{i}}{4}[\gamma_\nu, \gamma_\lambda]\left(\frac{1-\gamma_5}{2}\right)(\partial_\mu\beta_\lambda + e\beta_\mu\beta_\lambda) + \frac{\mathrm{i}}{8}[\gamma_\mu, \gamma_\nu]\left(\frac{1-\gamma_5}{2}\right)e(\beta^2 + \tfrac{1}{4}\Lambda^2 + \tfrac{1}{4}\Omega^2) \\ &\quad \left. + \tfrac{1}{2}(\Lambda + \mathrm{i}\gamma_5\Omega)(b_\mu\gamma_\nu + \tfrac{1}{2}\epsilon_{\mu\nu\lambda\tau}b_\lambda\gamma_\tau)\right] - [\mu, \nu]\end{aligned} \tag{5}$$

where $[\mu, \nu]$ denotes the previous terms antisymmetrised in μ and ν.

The self-duality condition† then leads to the following equations for θ, ψ, Λ and Ω:

$$\Box\theta + \tfrac{1}{2}(\Lambda^2 + \Omega^2)\theta = 0 \tag{6}$$

$$\delta_{\mu\nu}\Box\psi - 4\partial_\mu\partial_\nu\psi = 0 \tag{7}$$

$$\partial_\mu\Lambda + e(a_\mu - 2b_\mu)\Lambda = 0 \tag{8}$$

$$\partial_\mu\Omega + e(a_\mu - 2b_\mu)\Omega = 0. \tag{9}$$

Before considering the full SO(6) case, we remark on the content of equations (6) and (7) for the SO(4) case, with $\Lambda = \Omega = 0$. It is then immediately obvious that we recover the t'Hooft (Jackiw *et al* 1977) and BPST (Belavin *et al* 1975) solutions from (6) and (7) respectively. We shall remark on this again below when we compute the topological invariants.

It follows from (8) and (9) that Λ and Ω are proportional, and without loss of generality we put $\Lambda = \Omega$. Integrating (8) and (9)

$$\Lambda = \Omega = \Omega_0\theta^{-1/2}\psi^{-3/2} \tag{10}$$

where ψ is found by integrating (7) to be

$$\psi = kx^2 + C_\mu x_\mu + D \tag{11}$$

where C_μ and D are constants. This can, without loss of generality, be re-expressed as

$$\psi = r^2 + \lambda^2 \tag{11'}$$

where $r^2 = x^2$, and the independent integration constants are λ^2 and Ω_0. The function $\theta(x)$ then obeys the Poisson equation

$$\Box\theta = -\Omega_0^2\psi^{-3}. \tag{6'}$$

† The SO(4) self-dual solutions with $b_\mu = 0$ result in a flat connection. The double self-dual solutions of SO(4) (Belavin *et al* 1975) or the self-dual solutions of SU(2) (Jackiw *et al* 1977) are non-trivial.

The general solution is then given by the special solution $\theta = \Omega_0^2/8\lambda^2\psi$ added to any solution of the Laplace equation:

$$\theta = A^2 + \frac{\Omega_0^2}{8\lambda^2\psi} + \frac{B^2}{r^2} + \theta_0(x) \tag{12}$$

where A^2 and B^2 are constants and $\theta_0(x)$ is the solution of the four-dimensional Laplace equation considered by Jackiw *et al* (1977)

$$\theta_0(x) = \sum_i \frac{\lambda_i^2}{|x - y_i|^2}.$$

There remains now to compute the Pontryagin charge

$$q = \frac{1}{16\pi^2} \int \mathrm{Tr}\, {}^*F_{\mu\nu}F_{\mu\nu}\, \mathrm{d}_4x.$$

The general expression for q for A_μ of the form (3) (or (5)) is

$$\begin{aligned} e^2q = \frac{1}{16\pi^2} \int \mathrm{d}_4x \Big\{ & [-\Box\Box \ln \theta - \tfrac{1}{2}\Box(\Lambda^2 + \Omega^2) + \nabla((\Lambda^2+\Omega^2)\nabla \ln \theta)] \\ & + (\Lambda^2+\Omega^2)\left[-\Box \ln \theta - \frac{3}{2}\left(\frac{\Box\psi}{\psi} - \left(\frac{\nabla\psi}{\psi}\right)^2\right) + \tfrac{1}{2}(\nabla \ln \theta^2\psi^3)^2 + \tfrac{1}{4}(\Lambda^2+\Omega^2)\right] \\ & + \frac{3}{2}\left[\frac{\Box\psi}{\psi} - 2\left(\frac{\nabla\psi}{\psi}\right)^2\right]^2 \Big\}. \end{aligned} \tag{13}$$

If now the equations of motion (6)–(9) are substituted all the terms in the integrand may be eliminated except the first and last leaving

$$e^2q = \frac{1}{16\pi^2} \int \left\{ -\Box\Box \ln \theta + \frac{3}{2}\left[\frac{\Box\psi}{\psi} - 2\left(\frac{\nabla\psi}{\psi}\right)^2\right]^2 \right\} \mathrm{d}_4x.$$

Substituting from (11) for ψ and integrating we obtain

$$e^2q = 1 - \frac{1}{16\pi^2} \int \Box\Box \ln \theta\, \mathrm{d}_4x. \tag{14}$$

If $\Omega_0 = 0$ we obtain the usual SO(4) result. It is easily seen that the solution (12) when $\Omega_0 \neq 0$ *has charge equal to the number of terms taken in addition to the term* $\Omega_0^2/8\lambda^2\psi$. In particular taking this last term alone gives a flat connection with zero charge. Taking

$$\theta = A^2 + (\Omega_0^2/8\lambda^2\psi)$$

gives a unit charge solution which is already in a non-singular gauge. If we take the most general spherically symmetric solution

$$\theta = A^2 + \frac{\Omega_0^2}{8\lambda^2\psi} + \frac{B^2}{r^2}$$

with A^2, Ω_0, B^2 all non-zero we get charge 2.

It should be observed that putting $\Lambda = \Omega$ to satisfy (8) and (9) actually reduces the *ansatz* (3) to an SO(5) *ansatz* since the last two terms can be grouped as

$$(1 + \mathrm{i}\gamma_5)\gamma_\mu\Omega/4e$$

and this together with $L_{\mu\nu}$ closes on an SO(5) algebra. Thus we have in fact found an SO(5) solution.

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